

Photon position observable

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In biorthogonal quantum mechanics, the eigenvectors of a quasi-Hermitian operator and those of its adjoint are biorthogonal and complete and the probability for a transition from a quantum state to any one of these eigenvectors is positive definite. We apply this formalism to the long standing problem of the position observable in quantum field theory. The dual bases are positive and negative frequency one-particle states created by the field operator and its conjugate and biorthogonality is a consequence of their commutation relations. In these biorthogonal bases the position operator is covariant and the Klein-Gordon wave function is localized. We find that the invariant probability for a transition from a one-photon state to a position eigenvector is the first order Glauber correlation function, bridging the gap between photon counting and the sensitivity of light detectors to electromagnetic energy density.

I. INTRODUCTION

Many applications and tests of quantum mechanics (QM) involve photons and some require a basis of photon position eigenvectors [1]. In spite of its potential for direct application to experiment, it has been concluded that there is no position observable or completeness relation for photons [2, 3]. In the assumed absence of photon number density, energy density is used to define the photon wave function as the positive frequency part of the Riemann-Silberstein vector $\mathbf{F} \propto \mathbf{E} \pm ic\mathbf{B}$ [4–7]. But a consistent one-particle theory and position observable does exist for Klein-Gordon (KG) particles [8, 9]. We will show here that this solution to the relativistic position observable problem can be extended to photons.

The KG field is sometimes used as a simple model of a photon, for example in curved space [10]. The zeroth component of the conventional KG four-current density that equals the difference between its particle and antiparticle parts leads to an indefinite scalar product unless the KG field is limited to positive frequencies. The positive frequency part of the four-current density is sometimes interpreted as particle number density but it can still be negative if components with two or more different frequencies are added [11–13]. However, if the KG scalar product is recognized as a pseudo-scalar product within the framework of pseudo-Hermitian QM [9], a positive definite particle density *is* obtained. This can be extended to an arbitrary linear combination of positive and negative frequency terms if a new scalar product is defined in which the particle and antiparticle contributions are added [8, 9, 14, 15]. Moreover, Hegerfeldt's theorem tells us that restriction to positive frequencies leads to instantaneous spreading of a particle's wave function [16],

so negative frequencies are needed to obtain a causally evolving particle density. A photon, like a neutral KG particle, is its own antiparticle so these advances in our understanding of KG particle density are very relevant to our goal here, which is to obtain analogous results for photons.

The KG wave function derived in [9] is the projection of a particle's state vector onto the eigenvectors of the Newton Wigner (NW) position operator [17]. NW found a position operator for KG particles, but they concluded that the only photon position operator is the Pryce operator whose vector components do not commute [18], making the simultaneous determination of photon position in all three directions of space impossible. They had assumed spherically symmetrical position eigenstates for photons, while photon position eigenvectors have an axis of symmetry like twisted light [19]. Following the NW method with omission of the spherical symmetry axiom, a photon position operator with commuting components and eigenvectors that are cylindrically symmetrical in \mathbf{k} -space can be constructed [20]. Since spin and orbital angular momentum are not separately observable [21], its eigenvectors have only definite total angular momentum along some fixed but arbitrary axis [22].

Here all calculations will be performed in the physical Hilbert space of solutions to the wave equation using a scalar product of the form derived in [9] and biorthogonal QM [24] summarized here as follows: The eigenvectors of a quasi-Hermitian [25] operator \hat{O} and its adjoint \hat{O}^\dagger are not orthogonal, as is the case for conventional Hermitian operators, but biorthogonal. This means that, given the eigenvector equations

$$\hat{O}|\omega_i\rangle = \omega_i|\omega_i\rangle, \quad (1)$$

$$\hat{O}^\dagger|\tilde{\omega}_j\rangle = \omega_j|\tilde{\omega}_j\rangle \quad (2)$$

we have $\langle\tilde{\omega}_j|\omega_i\rangle = \delta_{ji}\langle\tilde{\omega}_i|\omega_i\rangle$ and the completeness relation $\hat{1} = \sum_i |\omega_i\rangle\langle\tilde{\omega}_i|/\langle\tilde{\omega}_i|\omega_i\rangle$. An arbitrary state $|\psi\rangle$ has an associated state $|\tilde{\psi}\rangle$. If an arbitrary state vector

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is expanded as $|\psi\rangle = \sum_i c_i |\omega_i\rangle$ in the Hilbert space \mathcal{H} then in biorthogonal QM its associated state is $|\tilde{\psi}\rangle = \sum_i c_i |\tilde{\omega}_i\rangle \in \mathcal{H}^*$ where $c_i = \langle \tilde{\omega}_i | \psi \rangle = \langle \omega_i | \tilde{\psi} \rangle$. Using these expansions it is straightforward to verify that $\langle \tilde{\psi}_1 | \psi_2 \rangle = \langle \psi_1 | \tilde{\psi}_2 \rangle$. The probability for a transition from a quantum state $|\psi\rangle$ to an eigenvector $|\tilde{\omega}_i\rangle$ of \hat{O}^\dagger is

$$p_i = \frac{|\langle \tilde{\omega}_i | \psi \rangle|^2}{\langle \tilde{\psi} | \psi \rangle \langle \tilde{\omega}_i | \omega_i \rangle}. \quad (3)$$

A generic operator can be written in the form

$$\hat{F} = \sum_{i,j} f_{ij} |\omega_i\rangle \langle \tilde{\omega}_j| \quad (4)$$

where f_{ij} can be viewed as a matrix [24]. In Section III we will apply this formalism to the biorthogonal position eigenvectors $|\phi(x)\rangle = \hat{\phi}(x)|0\rangle$ and $|\tilde{\phi}(x)\rangle = |\pi(x)\rangle = \hat{\pi}(x)|0\rangle$ where $x^\mu = (ct, \mathbf{x})$, $\hat{\phi}(x)$ is a field operator, $\hat{\pi}(x)$ is its conjugate momentum operator, and $|0\rangle$ is the vacuum state.

The rest of this paper is organized as follows: In Section II KG wave mechanics, with the field rescaled here to facilitate application to particles with zero mass, is reviewed. In Section III biorthogonality of the one-particle states created by the field operator and its conjugate momentum are examined and the covariant position operator and positive definite probability density are derived. A second quantized formalism is used to facilitate future application to multiparticle problems such as entanglement. In Section IV the KG position observable discussed in Sections II and III is extended to photons. In Section V the wave function of the photon emitted by an atom is discussed and in Section VI we conclude.

The configuration space scalar field and four-potential will be called $\phi(x)$ and $A^\mu(x)$. State vectors such as $|\psi(t)\rangle$ and position eigenvectors such as $|\phi(x)\rangle$ and $|\mathbf{E}_\lambda(x)\rangle$ introduced in Sections III and IV are given as expansions in Fourier space. The function $\psi(x)$ used in [9, 27] to represent the scalar field will be reserved for the wave function that equals the projection of a particle's state vector onto a basis of position eigenvectors. The KG and photon wave functions, $\psi(x)$ and $\psi_\perp(x)$ respectively, are proportional to probability amplitudes, with units that differ from those of $\phi(x)$, $A^\mu(x)$ and their spacetime derivatives.

II. KLEIN-GORDON WAVE MECHANICS

We will start with a review of the KG position observable problem. The KG equation

$$\partial_\mu \partial^\mu \phi(x) + \frac{m^2 c^2}{\hbar^2} \phi(x) = 0 \quad (5)$$

describes charged and neutral particles with zero spin (pions). Here covariant notation and the mostly minus convention are used in which $x^\mu = x = (ct, \mathbf{x})$, $\partial_\mu = (\partial_{ct}, \nabla)$, m is the mass of the KG particle, c is the speed of light, $2\pi\hbar$ is Planck's constant and $f_1 \overleftrightarrow{\partial}_\mu f_2 \equiv f_1 (\partial_\mu f_2) - (\partial_\mu f_1) f_2$. The function $\phi(x)$ is any scalar field that satisfies the KG equation (5). The four-density

$$J_{KG}^\mu(x) = ig\phi(x)^* \overleftrightarrow{\partial}^\mu \phi(x), \quad (6)$$

satisfies a continuity equation. Plane wave normal mode solutions to (5) proportional to $\exp(-i\omega t)$ are referred to as positive frequency solutions, while those proportional to $\exp(i\omega t)$ are negative frequency. Completeness requires that both positive and negative frequency modes be included. Their contributions to $J_{KG}^0(x)$ are of opposite sign, so $J_{KG}^0(x)$ is interpreted as charge density and the quantity g in (6) is set equal to qc/\hbar for particles of charge q .

If only particles, as opposed to both particles and antiparticles, are to be considered, then the KG field can be restricted to positive frequencies and the scalar product [10]

$$(\phi_1, \phi_2)_{KG} = \frac{i}{\hbar} \int_t d\mathbf{x} \phi_1(x)^* \overleftrightarrow{\partial}_t \phi_2(x) \quad (7)$$

is positive definite. Here t denotes a spacelike hyperplane of simultaneity at instant t . The integrand of (7) looks like a particle density but this is misleading since, as noted in the Introduction, $J_{KG}^0(x)$ is not positive definite for positive frequency fields.

The problem of a probability interpretation for KG particles has a long history. Lack of a probability interpretation led Dirac to derive his celebrated equation for spin half particles, but this does not solve the problem for KG fields. In a seminal paper intended to clarify the confusion about relativistic wave mechanics, Feshbach and Villars reviewed the two component formalism that separates the wave function into its particle and antiparticle parts for charged or for neutral particles [27]. Since then various strategies have been employed to derive a positive definite probability density. The four-current density can be redefined so that its zeroth component is positive definite [28], but this construction has no apparent physical basis and it fails if $m = 0$ [29]. It has been proposed that for charged pions only positive definite eigenstates of the Hamiltonian are physical [30]. A new $J^\mu(x)$ was derived that does not require separation of the field into positive and negative frequency parts [31] so it can be applied to the real fields that describe neutral pions. If $\phi(x)$ is restricted to positive frequencies it reduces to (6) so this $J^\mu(x)$ that describes an arbitrary linear combination of positive and negative frequency fields, including real fields, will be used here.

Working in the two component formalism with a pseudo-Hermitian Hamiltonian Mostafazadeh [9] defined the positive-definite Hermitian operator

$$\hat{D} \equiv -\nabla^2 + m^2 c^2 / \hbar^2 \quad (8)$$

in terms of which the KG equation is $(\hat{D} + \partial_{ct}^2)\psi = 0$. He derived the conjugate field

$$\phi_c(x) = i\hat{D}^{-1/2}\partial_{ct}\phi(x) \quad (9)$$

such that if $\phi = \phi^+ + \phi^-$ then $\phi_c = \phi^+ - \phi^-$. This implies that ϕ_c is a scalar. It is then straightforward to verify that ϕ_c satisfies the KG equation and that

$$J^\mu(x) = \frac{i}{\hbar}\phi(x)^* \overleftrightarrow{\partial}^\mu \phi_c(x) \quad (10)$$

satisfies the continuity equation $\partial_\mu J^\mu = 0$. Up to a constant that just scales J^μ , (10) is the expression derived in [31]. Like (6) $J^\mu(x)$ is manifestly covariant. It was proved in [9, 31] that the scalar product

$$(\phi_1, \phi_2) = \frac{i}{\hbar} \int_t d\mathbf{x} \phi_1(x)^* \overleftrightarrow{\partial}_t \phi_2(x) \quad (11)$$

is positive definite, time independent and can be written in covariant form as

$$(\phi_1, \phi_2) = \frac{i}{\hbar} \int_n d\sigma n_\mu \phi_1(x)^* \overleftrightarrow{\partial}^\mu \phi_2(x) \quad (12)$$

where n is an arbitrary spacelike hyperplane with normal n^μ , in other words, it is a Cauchy surface. The infinitesimal volume elements $d\sigma \equiv d\mathbf{x}$ are invariant. When restricted to positive frequencies (10) reduces to (6) and (11) reduces to (7). Using (5), (9) and $\phi_c = \phi^+ - \phi^-$, the scalar product (11) can be written as

$$(\phi_1, \phi_2) = \frac{2c}{\hbar} \sum_{\epsilon=\pm} \langle \phi_1 | \hat{D}^{1/2} \phi_2^\epsilon \rangle \quad (13)$$

where

$$\langle \chi_1 | \chi_2 \rangle = \int d\mathbf{x} \chi_1^*(\mathbf{x}) \chi_2(\mathbf{x}). \quad (14)$$

The non-relativistic Hilbert space is the vector space of square integrable continuous functions with the scalar product (14). In the relativistic Hilbert space the scalar product used here is (13). These scalar products can be evaluated in configuration space or in \mathbf{k} -space. The covariant Fourier transform is

$$\phi^\epsilon(x) = \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \pi^\epsilon(\mathbf{k}) e^{-i\epsilon(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})}. \quad (15)$$

Since $\phi^\epsilon(x)$ is a scalar and $d\mathbf{k} / [(2\pi)^3 2\omega_{\mathbf{k}}]$ is invariant, $\pi^\epsilon(\mathbf{k}) \in \mathcal{H}^*$ is a scalar, analogous to the transformation properties of photons [23]. For $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 c^2 + m^2 c^4 / \hbar^2}$ the function $\phi^\epsilon(x)$ satisfies the KG equation and (13) can be written as

$$(\phi_1, \phi_2) = \frac{1}{\hbar} \sum_{\epsilon=\pm} \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \pi_1^{\epsilon*}(\mathbf{k}) \pi_2^\epsilon(\mathbf{k}). \quad (16)$$

In the biorthogonal formalism the bases $\{\phi_{\mathbf{x}_j}^\epsilon(\mathbf{k})\} = \{\pi_{\mathbf{x}_j}^\epsilon(\mathbf{k}) / \omega_{\mathbf{k}}\} \in \mathcal{H}$ and $\{\pi_{\mathbf{x}_j}^\epsilon(\mathbf{k})\} \in \mathcal{H}^*$ are biorthogonal and complete and the Hermitian adjoint of an operator is its complex conjugate transpose [24]. Since the scalar product (13) is positive definite, standard QM can be recovered if a nontrivial metric operator $\langle \cdot | \Theta \cdot \rangle$ is introduced [26]. In this metric formulation the basis is $\{\pi_{\mathbf{x}_j}^\epsilon(\mathbf{k})\}$ and operators are Hermitian. With the flat metric $\langle \cdot | \cdot \rangle$ operators representing observables can be non-Hermitian with biorthogonal eigenvectors. The norm and orthogonality of the elements of the Hilbert space and the concept of Hermiticity are determined by the definition of scalar product. Newton and Wigner defined the KG fields $\pi_{\mathbf{x}_j}^\epsilon(\mathbf{k}) \propto e^{-i\mathbf{k}\cdot\mathbf{x}_j} \omega_{\mathbf{k}}^{1/2}$ that satisfy $(\phi_1, \phi_2) \propto \delta(\mathbf{x}_1 - \mathbf{x}_2)$ and are eigenvectors of the position operator $i\nabla_{\mathbf{k}} - \frac{ic^2}{2\omega_{\mathbf{k}}} \mathbf{k}$ [17]. Hermiticity of the NW position operator with eigenvectors of this form is discussed by Pike and Sarkar [32]. The NW position operator has played a central role in the discussion relativistic particle position since its publication in 1949 [17], but this operator is not covariant and its eigenvectors are not localized. We will show in the next section that the covariant commutation relations are consistent with (13) and that the formalism of biorthogonal QM leads to a covariant position operator. A second quantized version of the biorthogonal formulation in [24] will be used to facilitate applications in quantum optics and understanding of the relationship of the classical wave equation to quantum field theory (QFT).

III. KG POSITION EIGENVECTORS

In QFT particles are created at a point in spacetime by a field operator or its canonical conjugate. The interaction picture (IP) scalar field operators $\hat{\phi}(x)$ and $\hat{\pi}(x) = \partial_t \hat{\phi}(x)$ will be written as

$$\hat{\phi}(x) = \sqrt{\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} e^{i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} \hat{a}^\dagger(\mathbf{k}) + \text{H.c.}, \quad (17a)$$

$$\hat{\pi}(x) = i\sqrt{\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3 2} e^{i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} \hat{a}^\dagger(\mathbf{k}) + \text{H.c.} \quad (17b)$$

where H.c. is the Hermitian conjugate and the covariant normalization condition is [33]

$$[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{q})] = (2\pi)^3 2\omega_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{q}). \quad (18)$$

On the t hyperplane the field operators satisfy the commutation relations

$$[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta(\mathbf{x} - \mathbf{y}). \quad (19)$$

If the vacuum state $|0\rangle$ is defined by the condition $\forall \mathbf{k} \hat{a}(\mathbf{k})|0\rangle = 0$ then the field operators create one-particle states in this vacuum. In the IP the basis vectors are time dependent [21]. To accomodate positive

and negative frequency wavefunctions, $\epsilon = \pm$ states will be defined as

$$|\phi^\epsilon(x)\rangle = \sqrt{\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} e^{i\epsilon(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} |1_{\mathbf{k}}\rangle, \quad (20a)$$

$$|\pi^\epsilon(x)\rangle = \sqrt{\hbar} \int \frac{d\mathbf{k}}{(2\pi)^3 2} e^{i\epsilon(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{x})} |1_{\mathbf{k}}\rangle, \quad (20b)$$

where $|\phi^+(x)\rangle \equiv \hat{\phi}^-(x)|0\rangle$ and $|\pi^\epsilon(x)\rangle \equiv c\hat{D}^{1/2}|\phi^\epsilon(x)\rangle$ so that the phase factor i is absorbed into the bases and

$$i\partial_t |\pi^\epsilon(x)\rangle = -\epsilon c\hat{D}^{1/2} |\pi^\epsilon(x)\rangle. \quad (21)$$

With these definitions $\langle\pi^+(x)|\psi^+\rangle$ is positive frequency while

$$\langle\pi^-(x)|\psi^-\rangle = \langle\pi^+(x)|\psi^+\rangle^* = \langle\psi^+|\pi^+(x)\rangle \quad (22)$$

is negative frequency where $\epsilon = +$ refers to a particle arriving from the past and absorbed on n , while $\epsilon = -$ refers to a particle emitted on n and propagating into the future. These basis vectors are biorthogonal in the sense that

$$\langle\pi^\epsilon(x)|\phi^{\epsilon'}(y)\rangle = \frac{\hbar}{2} \delta_n(x-y) \delta_{\epsilon\epsilon'}. \quad (23)$$

The notation $\delta_n(x-y)$ is defined to select x and y such that $x^\mu = y^\mu$ on the hyperplane with normal n_μ . Since $|\phi^\epsilon(x)\rangle$ and $|\pi^\epsilon(x)\rangle$ are biorthogonal, they satisfy the completeness relation

$$\hat{1} = \frac{2}{\hbar} \sum_{\epsilon=\pm} \int d\mathbf{x} |\phi^\epsilon(x)\rangle \langle\pi^\epsilon(x)| \quad (24)$$

where the factor $2/\hbar$ is due to normalization (see (23)).

There is a direct correspondence between the scalar product (13) and the vacuum expectation value of the QFT commutator. Substitution of (20b), (21) and (22) in the vacuum expectation value of (19) gives

$$\begin{aligned} \langle\phi(x)|\pi(y)\rangle &= \langle 0 | \hat{\phi}^+(x) \hat{\pi}^-(y) - \hat{\pi}^+(y) \hat{\phi}^-(x) | 0 \rangle \\ &= \langle\phi^+(x)|\pi^+(y)\rangle + \langle\pi^+(y)|\phi^+(x)\rangle \\ &= \langle\phi^+(x)|\pi^+(y)\rangle + \langle\phi^-(x)|\pi^-(y)\rangle. \end{aligned} \quad (25)$$

On the t hyperplane events $x^\mu = (ct_x, \mathbf{x})$ and $y^\mu = (ct_y, \mathbf{y})$ appear simultaneous but an inertial observer with velocity $c\beta$ will see these events as time ordered. Since the Fourier space integrand of $\langle\phi^+(x)|\pi^+(y)\rangle$ is proportional to $e^{-i\omega_{\mathbf{k}}(t_x - t_y)}$ while that of $\langle\pi^-(y)|\phi^-(x)\rangle$ will be seen as proportional to $e^{i\omega_{\mathbf{k}}(t_x - t_y)}$, if $t_x > t_y$ the first term is positive frequency ($\epsilon = +$) while the second is negative frequency ($\epsilon = -$). This assignment is not unique, since an observer with velocity $-c\beta$ will see the opposite time order. Thus the covariant scalar product (25) should be a sum over $\epsilon = \pm$, consistent with (13) but inconsistent with (7). Based on (25) or (12) a particle is either emitted or absorbed at x on n and there are no $\epsilon = +/\epsilon = -$ cross terms in the scalar product.

It can be verified by substitution that the basis states (20) are eigenvectors of a position operator of the form (4),

$$\hat{\mathbf{x}} = \frac{2}{\hbar} \sum_{\epsilon=\pm} \int d\mathbf{x} \mathbf{x} |\phi^\epsilon(x)\rangle \langle\pi^\epsilon(x)|, \quad (26)$$

and its adjoint, that is

$$\hat{\mathbf{x}} |\phi^\epsilon(x)\rangle = \mathbf{x} |\phi^\epsilon(x)\rangle, \quad (27a)$$

$$\hat{\mathbf{x}}^\dagger |\pi^\epsilon(x)\rangle = \mathbf{x} |\pi^\epsilon(x)\rangle, \quad (27b)$$

consistent with their biorthogonality. Any one-particle state can therefore be projected onto the position bases as

$$\begin{aligned} |\psi(t)\rangle &= \hat{1} |\psi(t)\rangle \\ &= \frac{2}{\hbar} \sum_{\epsilon=\pm} \int d\mathbf{x} |\phi^\epsilon(x)\rangle \langle\pi^\epsilon(x)|\psi(t)\rangle, \end{aligned} \quad (28a)$$

$$\begin{aligned} |\tilde{\psi}(t)\rangle &= \hat{1} |\tilde{\psi}(t)\rangle \\ &= \frac{2}{\hbar} \sum_{\epsilon=\pm} \int d\mathbf{x} |\pi^\epsilon(x)\rangle \langle\phi^\epsilon(x)|\tilde{\psi}(t)\rangle. \end{aligned} \quad (28b)$$

The wave function

$$\psi^\epsilon(x) = \langle\pi^\epsilon(x)|\psi(t)\rangle \quad (29)$$

completely describes the one-photon state $|\psi(t)\rangle$ in the $\{|\phi^\epsilon(x)\rangle\}$ basis of position eigenvectors. It may have positive frequency and negative frequency components. According to the rules of biorthogonal QM outlined in Section I we have the equality

$$\langle\phi^\epsilon(x)|\tilde{\psi}(t)\rangle = \langle\pi^\epsilon(x)|\psi(t)\rangle. \quad (30)$$

Using (23), (28) and (30) the squared norm of $|\psi(t)\rangle$,

$$\langle\psi|\psi\rangle = \frac{2}{\hbar} \sum_{\epsilon=\pm} \int d\mathbf{x} |\langle\pi^\epsilon(x)|\psi^\epsilon(t)\rangle|^2, \quad (31)$$

and the probability density,

$$p^\epsilon(x) = \frac{2}{\hbar \langle\psi|\tilde{\psi}\rangle} |\langle\pi^\epsilon(x)|\psi(t)\rangle|^2, \quad (32)$$

are positive definite. It is the invariant $p(x)$ given by (32), not the zeroth component of the four-current, $J^0(x)$, that describes particle density.

Since this application of biorthogonal QM is based on an invariant scalar, product the QM that it describes is covariant. In particular, the wave function of a plane wave, $|1_{\mathbf{k}}\rangle$, is $\langle 1_{\mathbf{k}} | 1_{\mathbf{q}} \rangle = (2\pi)^3 2\omega_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{q})$ in Fourier space and $\langle\phi(x)|1_{\mathbf{q}}\rangle = \sqrt{\hbar} e^{-i(\omega_{\mathbf{k}}t - \mathbf{q}\cdot\mathbf{x})}$ in configuration space and a localized state, $|\phi(y)\rangle$, is $\langle 1_{\mathbf{k}} | \phi(y) \rangle = \sqrt{\hbar} e^{i(\omega_{\mathbf{k}}t - \mathbf{k}\cdot\mathbf{y})}$ in Fourier space and $\langle\pi(x)|\phi(y)\rangle =$

$\frac{\hbar}{2}\delta_n(x-y)$ in configuration space. Eqs. (24) and (26) can be generalized to

$$\hat{1} = \frac{2}{\hbar} \int d\sigma |\phi(x)\rangle \langle -i\epsilon n_\mu \partial^\mu \phi(x)|, \quad (33)$$

$$\hat{x}_i = \frac{2}{\hbar} \int d\sigma x_i |\phi(x)\rangle \langle -i\epsilon n_\mu \partial^\mu \phi(x)| \quad (34)$$

respectively where $x_\mu n^\mu = ct_0$ on the hyperplane with normal n_μ and t_0 is the hyperplane of simultaneity [15, 34]. The matrix representing the position observable in configuration space is

$$\hat{x}_i = \langle \pi(x) | \hat{x}_i | \phi(y) \rangle_n = \frac{\hbar}{2} x_i \delta_n(x-y) \quad (35)$$

where x_i is on the n hyperplane.

The relationship of the relativistic position operator to the nonrelativistic position operator $i\nabla_{\mathbf{k}}$ and the NW position operator can be seen by transforming to Fourier space. The position operator (26) is in the IP while the conventional position operator is time independent so it is in the Schrödinger picture (SP). The unitary time evolution operator that transforms states and operators from the SP to the IP is, from (21), $\hat{U}(t) = \exp(-i\epsilon\hat{D}^{1/2}t)$. Using $\hat{\mathbf{x}}^{\text{SP}} = \hat{U}\hat{\mathbf{x}}\hat{U}^\dagger$ and expanding in the biorthogonal bases as in (4), the positive frequency part of the SP position operator (26) is

$$\hat{\mathbf{x}}^{\text{SP}+} = \int \frac{d\mathbf{k}}{2(2\pi)^3} \int \frac{d\mathbf{q}}{2(2\pi)^3} \int d\mathbf{x} \mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \left| \frac{1_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right\rangle \langle 1_{\mathbf{q}} | e^{i\mathbf{q}\cdot\mathbf{x}}. \quad (36)$$

Since $\int d\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} i\nabla_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} = 2(2\pi)^3 i\nabla_{\mathbf{k}} \delta(\mathbf{q}-\mathbf{k})$ [39],

$$\hat{\mathbf{x}}^{\text{SP}+} = \int \frac{d\mathbf{k}}{2(2\pi)^3} \left| \frac{1_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right\rangle i\nabla_{\mathbf{k}} \langle 1_{\mathbf{k}} |, \quad (37)$$

$$\hat{\mathbf{x}}^{\text{SP}+}(\mathbf{k}) = i\nabla_{\mathbf{k}}, \quad (38)$$

so that $i\nabla_{\mathbf{k}}$ is the position operator in the $|1_{\mathbf{k}}/\omega_{\mathbf{k}}\rangle \langle 1_{\mathbf{k}}|$ basis. When operating on $e^{-i\mathbf{k}\cdot\mathbf{x}}$ it extracts the position \mathbf{x} where the particle was created. With positive definite Hermian metric Θ the scalar product is $\langle \cdot | \Theta \cdot \rangle$ and an operator \hat{O} is quasi-Hermitian if $\Theta\hat{O}^\dagger = \hat{O}\Theta$. Since according to (26) $\hat{\mathbf{x}}^\dagger = \hat{D}^{1/2}\hat{\mathbf{x}}\hat{D}^{-1/2}$, the position operator is quasi-Hermitian [26]. Defining $S \equiv \Theta^{1/2}$, the operator $\hat{o} = \hat{o}^\dagger = S^{-1}\hat{O}^\dagger S$ is Hermitian [26]. In (16) the metric is $\Theta = \omega_{\mathbf{k}}^{-1}$ so $S = \omega_{\mathbf{k}}^{-1/2}$, the basis is $\left\{ \omega_{\mathbf{k}}^{-1/2} |1_{\mathbf{k}}\rangle \right\}$ and the matrix representing the NW position operator [9, 17, 31],

$$\hat{\mathbf{x}}_{\text{NW}}^{\text{SP}}(\mathbf{k}) = \omega_{\mathbf{k}}^{1/2} i\nabla_{\mathbf{k}} \omega_{\mathbf{k}}^{-1/2}, \quad (39)$$

is of the form $\hat{o}^\dagger = S^{-1}\hat{O}^\dagger S$ for $\hat{O}^\dagger = \hat{\mathbf{x}}^{\text{SP}+}(\mathbf{k})$ where $\omega_{\mathbf{k}}^{1/2} i\nabla_{\mathbf{k}} \omega_{\mathbf{k}}^{-1/2} = i\nabla_{\mathbf{k}} - \frac{i\epsilon^2}{\omega_{\mathbf{k}}} \mathbf{k}$. The factors $\omega_{\mathbf{k}}^{\pm 1/2}$ introduce nonlocality into the configuration space description

of the position eigenvectors. These nonlocal factors also appear in the expressions for the field operators in quantum optics [3, 21, 22], but we see here that this nonlocality is not physical since it does not appear in the covariant description of the position observable.

IV. PHOTONS

For photons the scalar field ϕ should be replaced with the four-vector potential A^μ . Both $A_\mu \partial^\nu A^\mu = (A_\mu \partial_{ct} A^\mu, A_\mu \nabla A^\mu)$ and $A_\nu F^{\nu\mu} = (\mathbf{A} \cdot \mathbf{E}/c, \mathbf{A} \times \mathbf{B})$ multiplied by $i\epsilon_0 c/\hbar$ are candidates for the four-current density, $F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu$ being the Faraday tensor. The properties of an operator of the form $iA_\nu F^{\nu\mu}$ were investigated in [40]. The Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$ is not Lorentz invariant, but A_μ can be chosen to transform as a Lorentz four-vector up to an extra term that maintains the Coulomb gauge in all frames of reference [41]. To avoid the complications associated with non-physical longitudinal and scalar photons the Coulomb gauge will be used here. In a source-free region in the Coulomb gauge both $A_\mu \partial^\nu A^\mu$ and $A_\nu F^{\nu\mu}$ reduce to $(\mathbf{A}_\perp \cdot \mathbf{E}_\perp/c, \mathbf{A}_\perp \times \mathbf{B})$.

Following (10) we can define

$$J^\mu(x) = \frac{i\epsilon_0 c}{\hbar} \sum_{\lambda,i} A_{\lambda i}(x)^* \overleftrightarrow{\partial}^\mu A_{\lambda i}(x) \quad (40)$$

where \mathbf{A}_λ is a transverse vector potential of helicity λ that satisfies the classical Maxwell wave equation $(\hat{D} + \partial_{ct}^2) \mathbf{A}_\lambda = 0$ where $m = 0$ so $\hat{D} = -\nabla^2$. The conjugate field is $\mathbf{A}_{\lambda c} \equiv i\hat{D}^{-1/2} \partial_{ct} \mathbf{A}_\lambda = \mathbf{A}_\lambda^+ - \mathbf{A}_\lambda^-$ where $\mathbf{A}_\lambda = \mathbf{A}_\lambda^+ + \mathbf{A}_\lambda^-$. It can then be verified by substitution that (40) satisfies the continuity equation $\partial_\mu J^\mu(x) = 0$. As a consequence the scalar product $\int_\sigma d\sigma n_\mu J^\mu(x)$ is Lorentz invariant. The photon scalar product that replaces (13) is

$$(A_1, A_2) = \frac{2\epsilon_0 c}{\hbar} \sum_{\lambda,\epsilon,i} \left\langle A_{1\lambda i}^\epsilon | \hat{D}^{1/2} A_{2\lambda i}^\epsilon \right\rangle \quad (41)$$

where $A = A^\mu = (0, \mathbf{A}_\perp)$.

For photons described in the Coulomb gauge, the field operator is $\hat{\mathbf{A}}_\perp(\mathbf{x}, t)$ and its canonical conjugate is $-\epsilon_0 \hat{\mathbf{E}}_\perp(\mathbf{x}, t)$ where $\hat{\mathbf{E}}_\perp = -\partial_t \hat{\mathbf{A}}_\perp$ is the electric field operator. The Fourier space spherical polar coordinates will be called k , $\theta_{\mathbf{k}}$ and $\phi_{\mathbf{k}}$ and their corresponding unit vectors $\mathbf{e}_{\mathbf{k}}$, \mathbf{e}_θ and \mathbf{e}_ϕ . The definite helicity transverse unit vectors are $\mathbf{e}_\lambda(\mathbf{k}) = (\mathbf{e}_\theta + i\lambda \mathbf{e}_\phi)/\sqrt{2}$ where λ is helicity, and the longitudinal unit vector is $\mathbf{e}_{\mathbf{k}}$. The NW photon position operator with commuting components can be written in Fourier space as [22] $\hat{\mathbf{x}} = \hat{E} (i\omega_{\mathbf{k}}^{1/2} \nabla_{\mathbf{k}} \omega_{\mathbf{k}}^{-1/2}) \hat{E}^{-1}$ where \hat{E} is a rotation through Euler angles to fixed reference axes. The basic idea is the same as was used in the derivation of the NW position operator in [9]; the position information is contained

in the factor $\exp(\mathbf{i}\mathbf{k} \cdot \mathbf{x})$ in the wave function but the factors $\omega_{\mathbf{k}}^{\pm 1/2}$ must be eliminated before the nonrelativistic position operator $i\nabla_{\mathbf{k}}$ can be used to extract this information. For the transverse fields that describe photons an additional unitary transformation \hat{E} that rotates the field vectors to axes fixed in space is needed. A position eigenvector has a vortex structure like twisted light [19, 22] with nonzero spatial extension in which the photon position eigenvalue \mathbf{x} is the center of internal angular momentum [34]. Here the NW-like position operator derived in [20] will not be used, but the definite helicity basis vectors $e_{\lambda}(\mathbf{k})$ that are defined for all \mathbf{k} are still needed to describe position eigenvectors.

Following the derivation in Section III, the IP photon position operator is

$$\hat{\mathbf{x}} = \frac{2}{\hbar} \sum_{\epsilon, \lambda=\pm} \int d\mathbf{x} \mathbf{x} |\mathbf{A}_{\lambda}^{\epsilon}(x)\rangle \cdot \langle \mathbf{E}_{\lambda}^{\epsilon}(x)|, \quad (42)$$

the position eigenvector equations are

$$\hat{\mathbf{x}} |\mathbf{A}_{\lambda}^{\epsilon}(x)\rangle = \mathbf{x} |\mathbf{A}_{\lambda}^{\epsilon}(x)\rangle, \quad (43a)$$

$$\hat{\mathbf{x}}^{\dagger} |\mathbf{E}_{\lambda}^{\epsilon}(x)\rangle = \mathbf{x} |\mathbf{E}_{\lambda}^{\epsilon}(x)\rangle \quad (43b)$$

and position basis states are

$$|\mathbf{A}_{\lambda}(x)\rangle \equiv \hat{\mathbf{A}}_{\lambda}^{-}(x) |0\rangle, \quad (44a)$$

$$|\mathbf{E}_{\lambda}(x)\rangle \equiv c\hat{D}^{1/2} \hat{\mathbf{A}}_{\lambda}^{-}(x) |0\rangle. \quad (44b)$$

Here the potential operator reads

$$\hat{\mathbf{A}}_{\lambda}^{-}(x) = \sqrt{\frac{\hbar}{\epsilon_0}} \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} \mathbf{e}_{\lambda}(\mathbf{k}) e^{-i(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t)} \hat{a}_{\lambda}^{\dagger}(\mathbf{k}). \quad (45)$$

The Fourier space canonical commutation relations and orthogonality relations are

$$[\hat{a}_{\lambda}(\mathbf{k}), \hat{a}_{\sigma}^{\dagger}(\mathbf{q})] = (2\pi)^3 2\omega_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{q}) \delta_{\lambda\sigma}, \quad (46a)$$

$$\langle 1_{\lambda, \mathbf{k}} | 1_{\sigma, \mathbf{q}} \rangle = (2\pi)^3 2\omega_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{q}) \delta_{\lambda\sigma}, \quad (46b)$$

where $|1_{\lambda, \mathbf{k}}\rangle \equiv a_{\lambda}^{\dagger}(\mathbf{k}) |0\rangle$.

With $\epsilon = \pm$ states defined as in Section III, the photon position eigenvectors are biorthogonal since their commutation relations imply that

$$\sum_{i=1}^3 \langle E_{\lambda i}^{\epsilon}(x) | A_{\sigma i}^{\epsilon'}(y) \rangle = \frac{\hbar}{2\epsilon_0} \delta_n(x - y) \delta_{\lambda\sigma} \delta_{\epsilon\epsilon'} \quad (47)$$

where the subscripts i denote Cartesian components of the three-vectors and $\epsilon = +$ for absorption at x while $\epsilon = -$ for emission at x . For these position eigenvectors the scalar product (41) is $(A_{\lambda}^{\epsilon}(x), A_{\sigma}^{\epsilon}(y)) = \delta_n(x - y) \delta_{\lambda\sigma}$. For free photons described by transverse fields the completeness relation is

$$\hat{1}_{\perp} = \frac{2\epsilon_0}{\hbar} \sum_{\epsilon, \lambda=\pm} \int d\mathbf{x} |\mathbf{A}_{\lambda}^{\epsilon}(x)\rangle \cdot \langle \mathbf{E}_{\lambda}^{\epsilon}(x)| \quad (48)$$

where we have defined

$$|\mathbf{A}_{\lambda}^{\epsilon}(x)\rangle \cdot \langle \mathbf{E}_{\lambda}^{\epsilon}(x)| \equiv \sum_{i=1}^3 |A_{\lambda i}^{\epsilon}(x)\rangle \langle E_{\lambda i}^{\epsilon}(x)|. \quad (49)$$

The identity operator $\hat{1}_{\perp}$ on the space of transverse photons is closely connected to the so-called ‘transverse Dirac delta’ of QED [21]. For any transverse one-photon state we can thence write

$$|\psi_{\perp}(t)\rangle = \frac{2\epsilon_0}{\hbar} \sum_{\epsilon, \lambda=\pm} \int d\mathbf{x} |\mathbf{A}_{\lambda}^{\epsilon}(x)\rangle \cdot \langle \mathbf{E}_{\lambda}^{\epsilon}(x) | \psi^{\epsilon}(t) \rangle \quad (50)$$

and the wave function

$$\psi_{\lambda}^{\epsilon}(x) = \langle \mathbf{E}_{\lambda}^{\epsilon}(x) | \psi^{\epsilon}(t) \rangle = \langle \mathbf{A}_{\lambda}^{\epsilon}(x) | \tilde{\psi}^{\epsilon}(t) \rangle \quad (51)$$

completely describes the one-photon state $|\psi_{\perp}(t)\rangle$ in either basis of position eigenvectors. The dual state vector is

$$|\tilde{\psi}_{\perp}(t)\rangle = \frac{2\epsilon_0}{\hbar} \sum_{\epsilon, \lambda=\pm} \int d\mathbf{x} |\mathbf{E}_{\lambda}^{\epsilon}(x)\rangle \cdot \langle \mathbf{A}_{\lambda}^{\epsilon}(x) | \tilde{\psi}^{\epsilon}(t) \rangle, \quad (52)$$

the squared norm is

$$\langle \psi_{\perp} | \tilde{\psi}_{\perp} \rangle = \frac{2\epsilon_0}{\hbar} \sum_{\epsilon, \lambda=\pm} \int d\mathbf{x} |\psi_{\lambda}^{\epsilon}(x)|^2 \quad (53)$$

and the *probability density* per photon for a transition from $|\psi_{\perp}(t)\rangle$ to the ϵ -frequency position eigenvector with helicity λ

$$\begin{aligned} p_{\lambda}^{\epsilon}(x) &= \frac{2\epsilon_0}{\hbar} \frac{|\psi_{\lambda}^{\epsilon}(x)|^2}{\langle \psi | \tilde{\psi} \rangle} \quad (54) \\ &= \frac{|\psi_{\lambda}^{\epsilon}(x)|^2}{\sum_{\epsilon, \lambda=\pm} \int d\mathbf{x} |\psi_{\lambda}^{\epsilon}(x)|^2} \end{aligned}$$

is positive definite.

A one-photon state can be Fourier expanded as

$$|\psi_{\perp}(t)\rangle = \sum_{\lambda, \epsilon=\pm} \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} c_{\lambda}^{\epsilon}(\mathbf{k}, t) |1_{\lambda, \mathbf{k}}\rangle. \quad (55)$$

Eqs. (44), (45) and (46b) give $\langle \mathbf{E}_{\lambda}(\mathbf{x}) | 1_{\lambda, \mathbf{k}} \rangle / \omega_{\mathbf{k}} = \langle \mathbf{A}_{\lambda}(\mathbf{x}) | 1_{\lambda, \mathbf{k}} \rangle$ so that $|1_{\lambda, \mathbf{k}} / \omega_{\mathbf{k}}\rangle \in \mathcal{H}$ and $|1_{\lambda, \mathbf{k}}\rangle \in \mathcal{H}^*$. Substitution in (51) then gives the dual state vector

$$|\tilde{\psi}_{\perp}(t)\rangle = \sum_{\lambda, \epsilon=\pm} \int \frac{d\mathbf{k}}{(2\pi)^3 2} c_{\lambda}^{\epsilon}(\mathbf{k}, t) |1_{\lambda, \mathbf{k}}\rangle. \quad (56)$$

The probability amplitude for a transition to a ϵ -frequency plane wave state with wave vector \mathbf{k} and helicity λ is proportional to $\langle 1_{\lambda, \mathbf{k}} | \psi^{\epsilon}(t) \rangle = \langle 1_{\lambda, \mathbf{k}} / \omega_{\mathbf{k}} | \tilde{\psi}^{\epsilon}(t) \rangle = c_{\lambda}^{\epsilon}(\mathbf{k}, t)$. According to the rules of

biorthogonal QM outlined in the Introduction the probability density per photon for this transition is

$$p_{\lambda}^{\epsilon}(\mathbf{k}) = \frac{|\langle 1_{\lambda,\mathbf{k}} | \psi^{\epsilon}(t) \rangle|^2}{\langle \tilde{\psi}(t) | \psi(t) \rangle (2\pi)^3 2} \\ = \frac{|c_{\lambda}^{\epsilon}(\mathbf{k}, t)|^2}{\sum_{\lambda, \epsilon=\pm} \int d\mathbf{k} |c_{\lambda}^{\epsilon}(\mathbf{k}, t)|^2}. \quad (57)$$

Time dependence of $c_{\lambda}^{\epsilon}(\mathbf{k}, t)$ indicates the presence of a source. When a photon is emitted by an atom, the expectation value of the photon number is smaller than one and approaches unity as $t \rightarrow \infty$. If $|\psi_{\perp}(t)\rangle$ is normalized so that $n(t) = \langle \tilde{\psi}(t) | \psi(t) \rangle$ is the number of photons, the probability density for $k^{\mu} = (\epsilon\omega_{\mathbf{k}}, \mathbf{k})$ is $|c_{\lambda}^{\epsilon}(\mathbf{k}, t)|^2 / [(2\pi)^3 2]$ while the probability density to find a photon at x on the hyperplane σ is $|\psi_{\lambda}^{\epsilon}(x)|^2 2\epsilon_0/\hbar$. In the SP the photon position operator (42) can be written as

$$\hat{\mathbf{x}}^{\text{SP}}(\mathbf{k}) = \hat{E} i \nabla_{\mathbf{k}} \hat{E}^{-1}. \quad (58)$$

The scalar product $\langle \mathbf{E}_{\lambda}^+(x) | \psi^+(t) \rangle = \langle \mathbf{E}_{\lambda}(x) | \psi(t) \rangle$ that leads to an invariant probability to count a photon is proportional to probability amplitude, not the electric field. Glauber defined an ideal photon detector as a system of negligible size with a frequency-independent photon absorption probability [35]. For the positive frequency one-photon state $|\psi(t)\rangle$ he found that the probability to count a photon is proportional to $|\langle \mathbf{E}_{\lambda}(x) | \psi(t) \rangle|^2$. Glauber considered photodetection to be a square law process and interpreted it to be responsive to the density of electromagnetic energy, but number density gives an invariant probability to count a photon while energy density does not. Indeed, the biorthogonal completeness relation (48) implies that a basis of ideal Glauber detectors can be defined provided the state vector $|\psi\rangle \in \mathcal{H}$ of the photon at hand has been created by the $\mathbf{A} \cdot \mathbf{p}$ minimal coupling Hamiltonian. In that case, $|\langle \mathbf{E}_{\lambda}(x) | \psi(t) \rangle|^2$ is proportional to photon probability density.

Here a positive definite particle density is obtained in the physical Hilbert space according to the rules of biorthogonal QM summarized in the Introduction. An alternative approach is to transform the physical fields to the Foldy representation [36] using the nonlocal operator $\hat{D}^{-1/4}$ and its inverse [9]. For photons this nonlocal transformation leads to the Landau-Peierls (LP) wave function [6, 38]. The disadvantage to this approach is that the relationship between the LP wave function and a current source is nonlocal. Here the fields $A^{\mu}(x)$ due to the local interaction Hamilton $j_{\mu}(x) A^{\mu}(x)$ are calculated first and the probability amplitude for a transition to a position eigenvector is then obtained using the invariant scalar product (41). These fields have well defined Lorentz and gauge transformation properties. The positive definiteness of the probability follows then directly from the mathematical rules of biorthogonal QM.

An advantage of the second quantized formalism used here is that multiphoton wave functions can be introduced as in [7, 35, 43]. For example, to the two photon state $|\psi_2\rangle$ can be associated the wave function

$$\psi_{\lambda_1, \lambda_2}(\mathbf{x}_1, \mathbf{x}_2, t) = \langle \mathbf{E}_{\lambda_1}(\mathbf{x}_1) \mathbf{E}_{\lambda_2}(\mathbf{x}_2) | \psi_2(t) \rangle \quad (59)$$

with $|\mathbf{E}_{\lambda_1}(\mathbf{x}_1) \mathbf{E}_{\lambda_2}(\mathbf{x}_2)\rangle \equiv \hat{\mathbf{E}}_{\lambda_1}(\mathbf{x}_1) \hat{\mathbf{E}}_{\lambda_2}(\mathbf{x}_2) |0\rangle$. This wave function localizes the photons at spatial points \mathbf{x}_1 and \mathbf{x}_2 at time t and can describe entangled two-photon states.

V. WAVE FUNCTION OF A PHOTON EMITTED BY AN ATOM

The wave function (51) for a photon emitted by a two-level atom initially in its excited state was derived in [5] and [41, 42], to first order in the IP minimal coupling interaction Hamiltonian $\hat{H}_I = (e/m_e) \hat{\mathbf{A}}(\hat{\mathbf{x}}_e, t) \cdot \hat{\mathbf{p}}_e(t)$. For a two-level atom initially in its excited state $|e\rangle$ with no photons present, the positive frequency IP wave function describing decay to its ground state $|g\rangle$ while emitting one photon is

$$|\psi(t)\rangle = c_e(t) |e, 0\rangle \\ + \sum_{\lambda=\pm} \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} c_{g,\lambda}(\mathbf{k}, t) |g, 1_{\lambda,\mathbf{k}}\rangle \quad (60)$$

where

$$c_{g,\lambda}(\mathbf{k}, t) = \frac{e}{m_e} \langle g, 1_{\lambda,\mathbf{k}} | \hat{\mathbf{A}}_{\lambda}^{-}(\hat{\mathbf{x}}_e, t) \cdot \hat{\mathbf{p}}_e(t) | e, 0 \rangle \\ \times \frac{1 - e^{i(\omega_{\mathbf{k}} - \omega_0)t}}{\hbar(\omega_{\mathbf{k}} - \omega_0)}. \quad (61)$$

Here $\hbar\omega_0$ is the level separation between the ground and excited states and $\hat{\mathbf{x}}_e$ and $\hat{\mathbf{p}}_e$ are the electron position and momentum operators. For $|\tilde{\psi}_{\perp}\rangle$ given by (56) the transverse single-photon state and its dual are thus given by

$$|\psi_{\perp}(t)\rangle = \sum_{\lambda=\pm} \int \frac{d\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}} c_{g,\lambda}(\mathbf{k}, t) |1_{\lambda,\mathbf{k}}\rangle, \quad (62a)$$

$$|\tilde{\psi}_{\perp}(t)\rangle = \sum_{\lambda=\pm} \int \frac{d\mathbf{k}}{(2\pi)^3 2} c_{g,\lambda}(\mathbf{k}, t) |1_{\lambda,\mathbf{k}}\rangle. \quad (62b)$$

In [41, 42] the minimal coupling Hamiltonian created the photon state vector $|\psi_{\perp}\rangle$ in the $|\mathbf{A}_{\lambda}(\mathbf{x})\rangle$ basis so the appropriate wave function is $\langle \mathbf{E}_{\lambda}(\mathbf{x}) | \psi_{\perp} \rangle$. Indeed we have from (51) and (60)

$$\psi_{\lambda}(x) = i\sqrt{\frac{\hbar}{\epsilon_0}} \int \frac{d\mathbf{k}}{(2\pi)^3 2} \mathbf{e}_{\lambda}(\mathbf{k}) e^{-ik_{\mu}x^{\mu}} c_{g,\lambda}(\mathbf{k}, t). \quad (63)$$

The positive frequency wave function of the emitted photon is calculated, but causal solutions that include negative frequencies are also considered. Taking into account

a factor $-i$ in the electric field operator used in [41, 42], $c_{g,\lambda} = -ic_{\lambda}^+$ in (55). Substitution of the wave function $\psi_{\lambda}(x) = \langle \mathbf{E}_{\lambda}(\mathbf{x}) | \psi_{\perp}(t) \rangle$ in (54) gives the probability density in space to count a photon at time t . Since in [41, 42] the wave function is normalized as $\langle \psi_{\perp} | \psi_{\perp} \rangle = 1$, the factor $\langle \psi_{\perp} | \tilde{\psi}_{\perp} \rangle$ in (54) and (57) approaches $1/\omega_0$ as $t \rightarrow \infty$.

If the standard (dipolar) $\mathbf{E} \cdot \mathbf{x}$ Hamiltonian were to be used instead, the photon would be created in the $|\mathbf{E}_{\lambda}(x)\rangle$ basis and the appropriate wave function would be $\langle \mathbf{A}_{\lambda}(x) | \tilde{\psi}_{\perp}(t) \rangle$.

VI. CONCLUSION

The formalism of biorthogonal systems can be, as we saw, called in action in relativistic quantum mechanics. It is particularly well-matched to the relativistic scalar product. In the biorthogonal formalism, both the Wigner-Bargmann quantum field operator (for photons, the vector potential) and its canonically conjugate momentum (for photons, the electric field) are put on an equal footing, and they generate respectively the direct and the dual basis of position eigenvectors of two different position operators, which are the Hermitian conjugate of each other. Our formalism further clarifies the meaning of the free parameter α [22, 41, 43] in the photon position operator. Here, the freedom in the choice of α allows us to use both \mathbf{x} and \mathbf{x}^{\dagger} .

The probability density (54) suggests a resolution of the apparent dichotomy between photon number counting and the sensitivity of a detector to energy density. The wave function $\langle \mathbf{E}_{\lambda}^+(x) | \psi(t) \rangle$ together with the state vector (60) describes creation of a photon in the time interval $0 \leq t' \leq t$ followed by its detection at time t . Since it is created in the $|\mathbf{A}_{\lambda}^{\epsilon}(x)\rangle$ basis and observed in the dual $|\mathbf{E}_{\lambda}^{\epsilon}(x)\rangle$ basis, that wave function is proportional to a probability amplitude. The probability density for a transition from $|\psi_{\perp}(t)\rangle$ to the position eigenvector at \mathbf{x} , given by $2\epsilon_0 |\langle \mathbf{E}_{\lambda}^+(x) | \psi(t) \rangle|^2 / \hbar$, is of the Glauber form [35]. However, in contrast to theories of photodetection based on energy density, we have proposed, through the position amplitude $\langle \mathbf{E}_{\lambda}^+(x) | \psi(t) \rangle$ in the dual basis, a true position measurement that describes an array of ideal photon counting detectors.

For a state vector that is an arbitrary linear combination of positive and negative frequency terms the probability density for a transition to the position eigenvector at \mathbf{x} is positive definite. This particle plus antiparticle probability density describes a particle at spatial location \mathbf{x} independent of whether it was absorbed or emitted. Thus (54) can be interpreted as probability density even if the wave function (51) is real as in classical electromagnetism. This application of biorthogonal QM is based on an invariant positive definite scalar product so transition probabilities are invariant and positive definite, the position operator is covariant, and there is no NW $\omega_{\mathbf{k}}^{\pm 1/2}$ nonlocality in the wave function.

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